REPRESENTATION OF GENERALIZED GROUPS BY FAMILIES OF BINARY RELATIONS

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ABSTRACT

Binary systems defined by families of binary relations satisfying special properties are studied. The existence of so called near-groups or quasiassociative loops is established.

Introduction. Notations follow those in the preceding papers [8, 9]. Here particular binary systems defined by families of binary relations with their generalized multiplication are studied.

§1 treats symmetric f.o.b.r., partitions of $N \times N$ (all $r \subset N \times N$), and f.o.b.r. satisfying certain combinations of conditions determining special categories of b.s. §2 establishes the existence of so called *near-groups* [7] or *quasiassociative loops* [3] which are not groups, although they are defined by a slight modification [5, 6] of Brandt's well-known *normal multiplication table* (n.m.t.) for groups.

The principal results of this paper were published without proof earlier [3]. Applications to groups are studied in [4].

1. Symmetric families of disjoint relations and quasiregular partitions.

A. *General properties*

All R are assumed to satisfy the conditions

$$(1)\qquad \begin{aligned} &r_1r_2 \neq \varnothing \Rightarrow \exists\, r_3 \in R \,|\, r_1r_2 \subset r_3 \\ &r_1 \neq r_2 \Rightarrow r_1 \cap r_2 = \varnothing \;(r_1, r_2 \in R) \end{aligned}$$

Every such R is *single valued* and satisfies

1. $r_a \cap d_N \neq \varnothing \Rightarrow a^2 = a$
2. a) $/r_a \cap /(r_e \cap d_N) \neq \varnothing \Rightarrow ea = a$
   b) $r_a/ \cap (r_e \cap d_N)/ \neq \varnothing \Rightarrow ae = a$
3. a) $x/x, x/y \in r_1,\ u/x \in r_2 \Rightarrow u/y \in r_2$
   b) $x/x, y/x \in r_1,\ x/u \in r_2 \Rightarrow y/u \in r_2$

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4. In a n.m.t. with two identical couples of corresponding rows and columns one couple can be deleted, without change of the b.s.

B. *Symmetric systems of relations* R_s

DEFINITION. R is *symmetric* and denoted R_s, if

$$r \in R \Rightarrow r^{-1} \in R \text{ (symmetry).}$$

By (1) it suffices to require

$$r \in R \Rightarrow (\exists r_1 \in R)\colon r^{-1} \subset r_1. \tag{2}$$

One denotes

$$r_a^{-1} = r_{a^{-1}}.$$

Clearly $(a^{-1})^{-1} = a$, aa^{-1} is always defined, and

$$aa^{-1} = e \Rightarrow e^2 = e,\ a^{-1}e = a^{-1},\ ea = a;$$

in particular, R_s has at least one idempotent element.

In R_s, A.3 generalizes to

$$(3)\qquad \begin{aligned} &x/y,\ x/z \in r_1,\ u/y \in r_2 \Rightarrow u/z \in r_2\\ &x/z,\ y/z \in r_1,\ x/u \in r_2 \Rightarrow y/u \in r_2 \end{aligned}$$

This and property A.4 imply that all repetitions of elements in a row or a column of a n.m.t. of R_s can be eliminated; without limitation of generality one may restrict oneself to R_s of 1-1 relations.

Further simple properties of an R_s:

$$a^2 = a \Rightarrow r_a \cap d_N \neq \varnothing,\ a^{-1} = a;$$

$$\exists\, b,\ ba = b \Rightarrow a^2 = a,\ b^{-1}b = a;\ \exists\, b,\ ab = b \Rightarrow a^2 = a,\ bb^{-1} = a;$$

$$a/b \in C,\ a^2 = a \Rightarrow ab = b;\ b/a \in C,\ a^2 = a \Rightarrow ba = b;$$

$$a/b \in C,\ a^2 = a,\ b^2 = b \Rightarrow a = b.$$

C. *Symmetric complete systems of relations*

R_c will denote a *complete* R, i.e.,

$$r_1, r_2 \in R_c \Rightarrow r_1 r_2 \neq \varnothing.$$

An R_c which is also an R_s will be denoted by R_{sc}.

THEOREM 1. *Every R_{sc} is an* IP *loop* (*loop with the inverse property*); *conversely, every* IP *loop can be represented by an R_{sc}.*

Proof. In an R_s, each of the six equalities

(4) $\quad ab = c,\ a^{-1}c = b,\ cb^{-1} = a,\ b^{-1}a^{-1} = c^{-1},\ c^{-1}a = b^{-1},\ bc^{-1} = a^{-1}$

implies the other five, as is obvious from the following extraction from the n.m.t.

$$\begin{array}{ccc} c & b & / \\ a & / & b^{-1} \\ / & a^{-1} & c^{-1} \end{array}$$

Consequently, R_{sc} is a quasigroup.

By the properties in the previous section, R_{sc} has an identity 1 ($r_1 = d_N$) and $aa^{-1} = a^{-1}a = 1$.

Thus R_{sc} is a loop.

Moreover, in an R_s

(5) $$a/b \in C \Rightarrow a^{-1}(ab) = b;\ \ b/a \in C \Rightarrow (ba)a^{-1} = b$$

In R_{sc}, ab and ba exist. (5) is thus satisfied for all a and b and the axioms of an IP loop are fulfilled.

Conversely, let L be an IP loop with identity 1. For each $(a,b,c) \in T_L (a,b,c \neq 1)$ construct the "subtable"

c	b	1
a	1	b^{-1}
1	a^{-1}	c^{-1}

The combination of all such subtables (e.g., by juxtaposition) into a larger table with the 1-s forming the diagonal, is a n.m.t. of an R_{sc} representing L.

D. *Quasiregular partitions*

Definition. An R constituting a partition of $N \times N$ (in brief N^2) will be called a *quasiregular partition* (q.p.) $Q = Q_N$ of N^2.

One denotes Q_c, Q_s, and Q_{sc}, respectively, a complete a symmetric, and a symmetric and complete q.p.

The simplest examples of q.p. are:

$Q^{(0)}$ — every element of N^2 is a relation.

$Q^{(1)}$ — the whole set N^2 is one relation.

Every Q contains idempotent elements, satisfies (3), and

(6) $\exists b \mid ba = b \Rightarrow a^2 = a;\ \exists b \mid ab = b \Rightarrow a^2 = a;\ a^2 = a \Rightarrow r_a \cap d_N \neq \varnothing.$

The lattice of the q.p.

The class $\mathfrak{Q}_N$ of all Q of N^2 constitutes a partially ordered system under $Q_1 \leqq Q_2$, Q_1 *finer* than Q_2, induced from the usual ordering of the class $\mathfrak{P}_N$ of all partitions of N^2.

$Q^{(0)} = P^{(0)}$ is the smallest and $Q^{(1)} = P^{(1)}$ the greatest member of $\mathfrak{Q}_N$ as well as of $\mathfrak{P}_N$.

LEMMA 1. *The greatest lower bound in* $\mathfrak{P}_N$ *of a family of* q.p. $R = \bigwedge \{Q_i\}_{i \in I}$ *is a* q.p.

Proof. $R = \{r_j\}$ is a partition. In order to show that this partition is quasi-regular one has to prove that $r_1 r_2 \cap r_3 \neq \varnothing \Rightarrow r_1 r_2 \subset r_3$.
Each $r_k = \bigcap_{i \in I} q_{ik}$ $(k = 1,2,3)$ where $q_{ik} \in Q_i$ is the relation of Q_i containing r_k. Therefore, for all i:

$$r_1 r_2 \subset q_{i1} q_{i2}$$

$$r_1 r_2 \cap r_3 \neq \varnothing \Rightarrow q_{i1} q_{i2} \cap q_{i3} \neq \varnothing \Rightarrow q_{i1} q_{i2} \subset q_{i3} \Rightarrow$$

$$\Rightarrow r_1 r_2 \subset \bigcap_{i \in I} q_{i3} = r_3.$$

THEOREM 2. $\mathfrak{Q}_N$ *is a complete lattice.*

Proof. $\mathfrak{P}_N$ is a complete lattice and $\mathfrak{Q}_N \subset \mathfrak{P}_N$.

$$P^{(1)} = Q^{(1)} \in \mathfrak{Q}_N.$$

The greatest lower bound of $\{Q_i\}$ in $\mathfrak{P}_N$ is $R \in \mathfrak{Q}_N$ (Lemma 1). Therefore, $\mathfrak{Q}_N$ is a complete lattice.

REMARK. $\mathfrak{Q}_N$ is *not* a sublattice of $\mathfrak{P}_N$. The following example shows that

$$Q_1 \overset{\mathfrak{P}_N}{\bigvee} Q_2 \subsetneqq Q_1 \overset{\mathfrak{Q}_N}{\bigvee} Q_2$$ (l.u.b. in $\mathfrak{P}_N$ and $\mathfrak{Q}_N$, respectively):

Q_1

b	c	e
a'	e	a''
e	a''	d

Q_2

b	c	e
a'	e	a''
e	a'	d

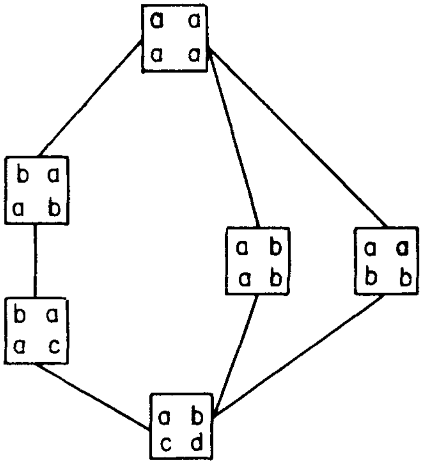
$P = Q_1 \overset{\mathfrak{P}_N}{\bigvee} Q_2$

b	c	e
a	e	a
e	a	d

Q_1 and Q_2 are q.p., but P is not a q.p. because $r_a r_a \cap r_d \neq \varnothing$ but $r_a r_a \not\subset r_d$.

The following examples show that unlike $\mathfrak{P}_N$, the lattice $\mathfrak{Q}_N$ is not semimodular.

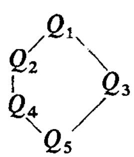
a) The lattice $\mathfrak{Q}_2$:



b) A sublattice of $\mathfrak{Q}_3$:

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad Q_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad Q_3 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix} \quad Q_4 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 3 & 2 \end{bmatrix}$$

$$Q_5 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 4 \\ 0 & 4 & 3 \end{bmatrix}$$



The q.p. Q_i represent the cyclic groups Z_i for $i = 1,2,3,4$, while Q_5 is a b.s. with multiplication not everywhere defined.

REMARK. Q_5 can be embedded in $Q \in \mathfrak{Q}_4$ representing Z_5:

$Q =$

3	2	1	0
2	1	0	4
1	0	4	3
0	4	3	2

DEFINITION. The b.s. M' is called a *homomorphic image* of the b.s. M under the mapping ϕ of M *onto* M', $\phi: |M| \to |M'|$, if

$$T_M\phi = T_{M'} \quad (T_M\phi = \{(a\phi, b\phi, c\phi)\}_{(a,b,c)\in T_M}).$$

ϕ is called a *homomorphism*. One writes $M' = M\phi$ and, therefore, $T_M\phi = T_{M\phi}$.
The homomorphic image of a single-valued b.s. is single-valued.

PROPOSITION 1. *$Q \leqq Q'$ (in $\mathfrak{Q}_N$) $\Rightarrow \exists\phi \,|\, Q' = Q\phi$ and ϕ is a homomorphism; i.e., Q' is a homomorphic image of Q. Conversely, every homomorphic image Q' of $Q(\in \mathfrak{Q}_N)$ is isomorphic to a* q.p. *of N^2.*

Proof. $Q \leqq Q' \Rightarrow (q \in Q \Rightarrow \exists! q' \in Q'$ such that $q \subset q')$. Denote by ϕ the mapping of Q onto Q' such that $q\phi = q'$.

$$(q_1, q_2, q_3) \in T_Q \Leftrightarrow q_1q_2 \subset q_3 \Rightarrow q_1'q_2' \cap q_3' \neq \varnothing \Leftrightarrow q_1'q_2' \subset q_3' \Leftrightarrow (q_1', q_2', q_3') \in T_{Q'},$$

i.e., $T_Q\phi \subset T_{Q'}$.

$$(q_1', q_2', q_3') \in T_{Q'} \Leftrightarrow q_1'q_2' \subset q_3' \Rightarrow \exists q_i \subset q_i' \; (i = 1,2,3) \,|\, q_1q_2 \cap q_3 \neq \varnothing \Leftrightarrow$$
$$\Leftrightarrow \exists (q_1, q_2, q_3) \in T_Q, \text{ i.e., } T_{Q'} \subset T_Q\phi.$$

Therefore, $T_Q\phi = T_{Q'}$.

Conversely, let ϕ be a homomorphism of Q and denote $Q' = Q\phi$. To every $q' \in Q'$ construct the 1-1 correspondence $q' \leftrightarrow p = \cup q'\phi^{-1}$. The relations p form a partition P of N^2.

$$p_1p_2 \neq \varnothing \Rightarrow \exists q_1 \subset p_1,\, q_2 \subset p_2 \,|\, q_1q_2 \neq \varnothing \Rightarrow \exists q_3 \,|\, (q_1, q_2, q_3) \in T_Q \Rightarrow$$
$$\Rightarrow (q_1', q_2', q_3') \in T_{Q'}$$

For every $q_i \in q_1'\phi^{-1}$, $q_j \in q_2'\phi^{-1}$ such that $q_iq_j \neq \varnothing$, the corresponding $q_k \supset q_iq_j$ must be mapped by the homomorphism ϕ onto q_3', i.e., $q_k \in q_3'\phi^{-1}$. Hence, $p_1p_2 = (\cup q_1'\phi^{-1})(\cup q_2'\phi^{-1}) \subset \cup q_3'\phi^{-1} = p_3$ and P is a q.p. of N^2.

At the same time one has shown that

$$p_1p_2 \neq \varnothing \Rightarrow (p_1, p_2, p_3) \in T_P \Rightarrow (q_1', q_2', q_3') \in T_{Q'}$$

and, conversely,

$$(q_1', q_2', q_3') \in T_{Q'} \Rightarrow (p_1, p_2, p_3) \in T_P,$$

i.e., $(p_1, p_2, p_3) \in T_P \Leftrightarrow (q_1', q_2', q_3') \in T_{Q'}$.

Therefore Q' is isomorphic to the q.p. P.

Various notions from group and loop theory can be transferred to q.p.; e.g., one can define a *direct product*.

$$Q_1 \times Q_2 \in \mathfrak{Q}_{N_1 \times N_2} \; (Q_i \in \mathfrak{Q}_{N_i},\, i = 1, 2).$$

2. Quasiassociative loops.

A. *Symmetric quasiregular partitions*

The class $\mathfrak{Q}_{sN}$ of the Q_s of N^2 is a complete lattice, sublattice of $\mathfrak{Q}_N$. The proof is similar to that for $\mathfrak{Q}_N$.

One verifies $Q_{sN_1} \times Q_{sN_2} = Q_{s,N_1 \times N_2}$.

A homomorphic image of a Q_s need not be a Q_s: every $Q \in \mathfrak{Q}_N$ is a homomorphic image of $Q^{(0)} \in \mathfrak{Q}_N$, which is clearly symmetric. (For examples of non-symmetric q.p. see p. 36).

For a $Q = \{q_i\}$ one defines its transpose $Q^* = \{q_i^{-1}\}$. Q^* is a q.p. anti-isomorphic to Q. Indeed, $q_i^{-1}q_j^{-1} \cap q_k^{-1} \neq \varnothing \Rightarrow q_jq_i \cap q_k \neq \varnothing \Rightarrow q_jq_i \subset q_k \Rightarrow q_i^{-1}q_j^{-1} \subset q_k^{-1}$.

Proposition 2. *For every Q, $U = Q \cap Q^*$ is the least fine symmetric* q.p. *finer than Q.*

Proof. a) U is a symmetric q.p.: As an intersection of two q.p. U is a q.p. and

$$u \in U \Rightarrow \exists i, j \mid u = q_i \cap q_j^{-1} \Rightarrow u^{-1} = q_i^{-1} \cap q_j \Rightarrow u^{-1} \in U.$$

b) Q is a homomorphic image of U (See Proposition 1.).

c) For every $S = S^*$, $U \leqq S \leqq Q$ implies also $U \leqq S \leqq Q^*$. Hence, $U \leqq S \leqq Q \cap Q^* = U \Rightarrow S = U$.

Every Q_{sc} must have a unique idempotent since every R_{sc} is an IP loop.

On the other hand, one has the

Proposition 3. *An associative Q with a unique idempotent is symmetric.*

Proof. Let e be the unique idempotent of the b.s. Q.

Assume q_a^{-1} intersects q_x and q_y, $x \neq y$. This implies $xa = e = ay$. By associativity

$$y = ey = (xa)y = x(ay) = xe = x.$$

Hence, $q_a^{-1} \subset q_x$ and by (2) Q is symmetric.

This proves:

Theorem 3. *A complete associative* q.p. *with a unique idempotent element is a group.*

B. *The existence of complete symmetric non-associative* q.p.

The symmetry of a Q_c implies the uniqueness of the idempotent element but not associativity, as can be seen from the following Q_{sc}, which is not a group†:

† In this connection, our thanks are due to Professor R. Artzy for drawing our attention to Cayley loops.

v	3	2	1	0
u	2	1	0	3
y	1	0	3	2
x	0	3	2	1
0	x	y	v	u

$$(xu)x = 2x = v$$
$$x(ux) = x1 = y$$

THEOREM 4. [3] *To every group (finite or infinite) G of order $n \geqq 5$ and to Z_4 there corresponds a Q_{sc}, which is not a group and which contains the group G as a normal subloop of index* 2.

Proof. *Case* 1: There exists in G an element a of order $\geqq 3$. Then three distinct elements of the form a, a^{-1}, $b(\neq e)$ can be found in G. One constructs the n.m.t. of G with the following initial section:

$\vdots$			
b			e
a^{-1}		e	
a	e		
e	a^{-1}	a	$\cdots$

Add to this table a new marginal column and a row identical with this column except for one transposition, as e.g.:

$\vdots$							
x_6						e	
x_5					e		
x_4	b			e			
x_3	a^{-1}		e				
x_2	a	e					
x_1	e	a^{-1}	a				
e	x_1	x_2	x_4	x_3	x_5	x_6	$\cdots$

This represents a Q_{sc} which is not associative:

a) Symmetry is maintained.

b) Completeness: All products of elements of G appear. An element x_i appearing in the first row multiplies from the left every x_j and the same element x_i appearing in the first column multiplies from the left all elements of G. Finally, any element $g \in G$ appears in every row (except the first) and thus multiplies from the left all elements of the first row, i.e., all x_i.

c) The table is a q.p.: the subtable of G is a q.p.; all multiplications involving x_i are performed only once except for the products of x_i with its symmetric elements which appear twice and are both equal e.

d) Non-associativity:

$$(x_1 a)x_1 = x_2 x_1 = a^{-1}$$

$$x_1(a x_1) = x_1 x_4 = b$$

Case 2. For every $g \in G$, $g^2 = e$. Then card $G \geqq 5$ implies that G contains a subgroup of type $Z_2 \times Z_2 \times Z_2$: $\{a, b, c, d, f, h, k, e\}$. One constructs, as above, a Q_{sc}, e.g.:

⋮								
x_7							e	
x_6	d	a	f	k	h	e		
x_5	f	k	d	a	e	h		
x_4	c	h	b	e	a	k		
x_3	h	c	e	b	d	f		
x_2	b	e	c	h	k	a		
x_1	e	b	h	c	f	d		
e	x_4	x_6	x_5	x_1	x_3	x_2	x_7	...

with

$$(x_4 x_3)x_2 = h x_2 = x_3$$

$$x_4(x_3 x_2) = x_4 k = x_6$$

In both cases G is a normal subloop of the loop represented by the Q_{sc}.

One can verify that for the groups Z_1, Z_2, Z_3 and V_4 these constructions give groups.

C. *Quasiassociative loops.*

DEFINITION. An IP loop that can be represented by a Q_{sc} will be termed a *quasiassociative loop* and denoted by QA.

LEMMA 2. (Condition I): *An* IP *loop* L *is a* QA *if and only if it contains a subset* $D = \{d_x, d_y, d_z, \cdots\}$ *(called a* generating column *of* L*) such that for every triplet* $d_x, d_y, d_z \in D$:

$$(d_x^{-1}d_y)(d_y^{-1}d_z) = d_x^{-1}d_z \tag{7}$$

and for every pair $f_1, f_2 \in L$ *there exists a triplet* $d_x, d_y, d_z \in D$ *such that*

$$f_1 = d_x^{-1}d_y,\ f_2 = d_x^{-1}d_z \tag{8}$$

Proof. Let Q_{sc} represent L. Denote by D the set of elements of the first column. Therefore, the elements of the first row form the set D^{-1}. The element of L on the place with coordinates $(x; y)$ will be denoted $f_{x;y}$ and put $f_{1;y} = d_y$. Then:

$$f_{x;1} = d_x^{-1}; f_{1;1} = d_1 = e; f_{x;y} = f_{x;1}f_{1;y} = d_x^{-1}d_y \tag{9}$$

The Q_{sc} represents a loop and for each couple $f_1, f_2 \in L$ there exists an element $f_3 \in L$ such that $f_1f_3 = f_2$, i.e., $f_1, f_2 \in L$ appear in a common column of the n.m.t., say $f_1 = f_{x;y}$ and $f_2 = f_{x;z}$. There exist, thus, $d_x, d_y, d_z \in D$ such that $f_1 = d_x^{-1}d_y$; $f_2 = d_x^{-1}d_z$ which proves (8).

By the definition of the multiplication in the Q_{sc} one has for every triplet $d_x, d_y, d_z \in D$:

$$(d_x^{-1}d_y)(d_y^{-1}d_z) = f_{x;y}f_{y;z} = f_{x;z} = d_x^{-1}d_z,$$

which proves (7).

Conversely, suppose that L has a generating column D. If $e \notin D$, $D \cup \{e\}$ too will satisfy (7) and (8). Assume, therefore, $e \in D$. Using (9) one constructs a n.m.t. with D as the first column and D^{-1} as the first row. All elements of the table are defined uniquely by:

$$f_{x;y} = d_x^{-1}d_y$$

The table is symmetric:

$$f_{y;x} = d_y^{-1}d_x = (d_x^{-1}d_y)^{-1} = f_{x;y}^{-1}$$

There are no inconsistencies in the table, because by (7):

$$f_{x;y}f_{y;z} = (d_x^{-1}d_y)(d_y^{-1}d_z) = d_x^{-1}d_z = f_{x;z}$$

By (8) there exists, for every $f_1, f_2 \in L$ a triplet $d_x, d_y, d_z \in D$ such that

$$f_1 = d_x^{-1}d_y, \qquad f_2 = d_x^{-1}d_z$$

It follows that every couple of distinct elements of L appears at least in one column, i.e., that all products in L exist in the n.m.t.

Every group is a QA ($D = G$ is permissible).

D can be used as an arbitrary column of the n. m. t. of QA; conversely, every column of such a table is a generating column, i.e., obeys (7) and (8).

D. IP *loops which are not* QA.

DEFINITION. Denote

$$C_r(a, b) = \{c \mid (ab)c = a(bc)\} \; (a, b, c \in L) \tag{10}$$

the set of elements associating at the right with the ordered pair $(a; b)$.

LEMMA 3. (Condition II): *Let L be an* IP *loop. If for some (a, b) $(a, b \in L)$, $C_r(a, b)$ does not contain a generating column, then L is* not *a* QA.

Proof. For every $r_a, r_b \in Q_{sc}$ representing L

$\exists x, y, z \in N$ such that $x/y \in r_a$, $y/z \in r_b$, hence

$z \in /r_c \Rightarrow r_a r_b r_c \neq \varnothing \Rightarrow c \in C_r(a, b)$.

The z column in the Q_{sc} can thus contain elements of $C_r(a, b)$ only. Therefore, if $C_r(a, b)$ does not contain a generating column, i.e., a subset that can serve as a column of the n.m.t. of L, then L is not a QA.

COROLLARY. *In every of the following cases the given* IP *loop L is not a* QA:

a) $\exists a, b, f_1, f_2 \in L \mid$ (there exists no $c_1, c_2, c_3 \in C_r(a, b) \mid f_1 = c_3^{-1}c_1; f_2 = c_3^{-1}c_2$).

b) $\exists a, b, f \in L \mid$ (there exists no $c_1, c_2 \in C_r(a, b) \mid f = c_1^{-1}c_2$)

(11) c) L' is a proper subloop of L and $\exists a, b \in L \mid C_r(a, b) \subset L'$

EXAMPLES: 1) Bruck [2, p. 33] constructs to each IP loop $L' = \{a, b, \cdots\}$ an IP loop $L = L' \cup \overline{L'} = \{a, b, \cdots; \bar{a}, \bar{b}, \cdots\}$ with multiplication defined by:

$$ab = ab \,(\text{in } L'); \; a\bar{b} = \overline{a^{-1}b}; \quad \bar{b}a = \overline{ba^{-1}}; \quad \bar{a}\bar{b} = b^{-1}a^{-1} \tag{12}$$

Apply this construction to groups L' with elements a, b such that $ab \neq ba$. *Then the corresponding L is not a* QA.

Proof. By (12): $(ab)\bar{c} = \overline{b^{-1}a^{-1}c}$; $a(b\bar{c}) = a\overline{b^{-1}c} = \overline{a^{-1}b^{-1}c}$. But $a^{-1}b^{-1} \neq b^{-1}a^{-1}$, hence $(ab)\bar{c} \neq a(b\bar{c})$ for every $\bar{c}$, and $C_r(a,b) = L'$. According to (11) L is not a QA.

2) *The smallest commutative Moufang loop* (*of order* 81) *is not a* QA.

Proof. One uses its construction (see [1]) as the set M of the 81 quadruples $A = (a_1, a_2, a_3, a_4)$ of elements of the prime field modulo 3. The operation in M is defined by:

$$AB = C \Leftrightarrow \begin{cases} c_i = a_i + b_i & (i = 1,2,3) \\ c_4 = a_4 + b_4 + (a_3 - b_3) \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} . \end{cases}$$

One computes the "associator" $A(BC) - (AB)C = \left[0,0,0, - \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \right]$.

One verifies that M is commutative and satisfies the Moufang identity $(AB)(CA) = [A(BC)]A$.

The subset $M' = \{(0, a_2, a_3, a_4)\}$ is a subloop of M, which is even a group. Let $A = (0,0,1,0)$, $B = (0,1,0,0)$. Then for any $X = (a_1, a_2, a_3, a_4)$ the associator

$$A(BX) - (AB)X = \left[0,0,0, - \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ a_1 & a_2 & a_3 \end{vmatrix} \right] = (0,0,0,a_1)$$

Hence, $X \in C_r(A,B) \Leftrightarrow X \in M'$ and by (11) M is not a QA.

The following properties are mentioned without proofs:

1. Every IP loop can be imbedded in a QA.
2. Every homomorphic image of a QA is a QA.
3. The direct product of two QA is a QA.

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